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SUBOPTIMIZATION MODEL OF DEMAND FOR INVESTMENT AND LABOR:AN OPTIMAL CONTROL THEORETIC APPROACH

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I. Introduction

Many empirical models of demand for capital goods (or for labor) introduce the dynamic relationship in the form of the stock adjustment pattern:

$$(1\text{-}1) \qquad I_t = \lambda[K_t^* - K_{t-1}]$$

where I_t, K_t^*, and K_{t-1} are, respectively, new investment at time t, desired stock of capital at time t, and actual stock of capital at time t-1. Equation (1-1) states that new investment will be proportional to the difference between desired and actual stock of capital. The unobservable quantity, K_t^*, is usually equated to some observable variables such as sales using relationship derived from the profit maximization principle with an explicit mathematical form imposed on the production function.(1)

As it stands equation (1-1) implies the geometrically declining distributed lags of Koyck [14] or of Nerlove [16]: the equation assumes

(1) Just as an illustration of this point, let us assume the Cobb-Douglas production function which has been the favorite among empiricists perhaps for its simplicity:

$$V_t = A\, L_t^{\alpha}\, K_t^{\beta}$$

where V_t, L_t, K_t are, respectively, quantity produced, labor employment, and capital stock at time t, and A, α, β are parameters. Then if perfect competition prevails, the marginal productivity condition for profit maximization will give rise to

$$K_t = \beta\, \frac{P_t V_t}{c_t}$$

where P_t and c_t are, respectively, price of V_t and the user cost. See Jorgenson [13].

that the maximum response of an economic variable occurs at the beginning of an adjustment period and the responses decline geometrically in the remaining periods. To get around this rigid assumption many writers [21, 12, 6, 1, 23] have suggested various distributed lag functions. These distributed lag functions, however flexible they may be, are applied to an equation such as (1-1) above in an _ad hoc_ fashion: the desired stock of capital, K_t^*, is often derived within the comparative static framework and just before the estimation the model is made dynamic by introducing some further assumptions, thus leaving unexplained the mechanism which gives rise to a particular distributed lag structure.(2)

Eisner and Strotz [10], for example, have derived the Koyck model such as equation (1-1) under appropriate functional forms with the stability of the desired stock imposed as an additional assumption.(3) In their paper they introduced a cost-of-expansion function (more commonly known as the cost of adjustment function) to allow for the situation in which the cost of investment may increase with the rate of expansion of capital stock, thus eliminating one of the shortcomings of the _ad hoc_ approach which does not incorporate any adjustment cost of investment in deriving the relationship involving the desired stock of capital, K_t^*. Since Eisner and Strotz published their paper, there have been a number of contributions to the theory of investment incorporating the costs of adjustment [15,11,22,4].

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(2) I am not implying here that this practice of _ad hoc_ introduction of distributed lag functions is in any way "inferior" to the other approaches such as the one used in this paper. Eisner and Strotz [10] show that if the interest rate is high, then the investment can be delayed; hence unimodal distributed lag coefficients with an inverse "J" shape are possible. Moreover in dealing with macro data, the process of aggregating data may give rise to distributed lags which may be well explained by some of the existing distributed lag functions. In an empirical study there is always an element of "_ad hocery_" and thus there may be cases in which these various distributed lag functions explain the given data better than the approach taken in this paper.

(3) The appropriate functional forms are that the profit function is quadratic in capital stock and that the adjustment cost function is quadratic in new investment.

All the optimal investment models treated in these articles assume an infinite time horizon, and some of them e.g. Lucas [15] and Treadaway [22] are concerned with long-run equilibrium values hence leaving little room for short-run consideration. For systems whose dynamics are deterministic and completely known, it will be reasonable to optimize the objective function over a long period of time. In a recent paper, however, Chetty [5] discusses the suboptimal adaptive controls based on subintervals of the time horizon, and he uses the technique of invariant imbedding equations of Bellman et. al. [3] to handle what is known as a two-point boundary value problem (TPBVP). In many practical situations the dynamics of the system may be complex and may undergo unforeseen changes or disturbances. Furthermore, the mathematical description of the system dynamics may be unknown. Then, we often find it convenient to use a model of assumed form which may be less complicated than the actual form and which will sequentially update information on the system dynamics. For this purpose the time horizon will be broken into subintervals and time will be treated as a running variable rather than as a fixed variable. Thus it can be seen that Chetty's approach may give practical insight to an econometrician who is always faced with these uncertainties.

The present paper is an attempt to extend Chetty's work. It derives a suboptimal adaptive control model for the demand for investment and labor by using invariant imbedding equations. In all the literature cited above, the optimal control problems are formulated by using a continuous maximum principle, and just before the estimation of a model derived from the maximum principle, it is changed to a descrete model. To an econometrician who uses discrete time series data, it will be better to formulate the optimal problem using a discrete maximum principle. Consequently, this paper deals with a discrete model.(4) Moreover an estimation procedure

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(4) As will be shown later the equations derived by the discrete maximum principle are similar (but not identical) to those derived by the continuous maximum principle. In computation the continuous nonlinear TPBVP is discretized, and equations resulting from this are similar but not the same as those obtained from the discrete nonlinear TPBVP problem. For a "small enough" sampling interval this may not be a serious problem. Moreover, the equations resulting from discrete invariant imbedding are in general the correct equations for the discrete model, whereas the discretization of the equations resulting from continuous invariant imbedding yields a a set of discrete equations whose solution is not the solution to the continuous problem. [See 18, pp.131-134; pp.531-540].

using instrumental variables is devised in this paper.

Section II presents a model of a firm and derives demand for investment and for labor using discrete imbedding equations. Section III discusses an estimation procedure and it has been applied to the data for the total Canadian manufacturing industries which are chosen as an example.

II. The Model

We conceive of the firm which sells at time t a product by the quantity, Q(t) at the price P(t). The firm employs two factors of production: capital, K(t) and labor, L(t), and the production function is given by F[K(t), L(t)]. Capital which is owned by the firm is accumulated at a gross real rate, I(t). The price of capital goods at time t is G(t). Labor is rented at the price W(t). Then the net cash flow of the firm, R(t), is given by

$$R(t) = P(t)Q(t) - G(t)I(t) - W(t)L(t). \tag{2-1}$$

Suppose that the firm plans to maximize the present value of net cash flow over the period from t_o to t_f:

$$V = \frac{1}{(1+\rho)^{t_f - t_o}} K(t_f)g(t_f) + \sum_{t=t_o}^{t_f-1} \frac{1}{(1+\rho)^t} R(t) \tag{2-2}$$

where the first term represents the discounted value of capital stock at time t_f, and ρ is the discount rate.

The Costs of Adjustment: As Eisner and Strotz [10] argue it is reasonable to think that on many occasions the cost of purchasing capital goods, C[I(t)],will tend to increase with plant expansion, and the question one faces is what constitutes the correct mathematical specification of the adjustment cost function. The specification of this function rests on empirical considerations, but the minimum conditions the function has to satisfy may be

$$C(I) > 0, \quad C'(I) > 0, \text{ and } C(0) = 0$$

and moreover the additional condition:

$$C''(I) > 0$$

will indicate that cost of adjustment increases with plant expansion at an accelerated rate. The simplest mathematical function which satisfies these conditions is

$$(2\text{-}3) \qquad G(t) = g(t) + \alpha I(t) \ , \quad \alpha > 0, \quad g(t) > 0$$

so that $C(I) = G(t)I(t)$ is a quadratic function of gross investment. The quadratic adjustment cost function is the one used by many authors [10, 15, 11, 4, 5].(5) Equation (2-3) consists of two parts: the first part, $g(t)$, is the price that would normally obtain if capital were a perfectly variable factor and if no inflexibilities of supply and installation of capital goods existed. The second part is included to account for these inflexibilities.

The classical treatment of labor as a purely variable factor in the short-run has been pointed out to be untenable in many practical situations and it has been argued that labor is a quasi-fixed factor [17, 9].(6) As

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(5) Stroz-Eisner [10], Lucas [15] and Treadaway [22], for example, make the cost of adjustment a function of <u>net</u> investment, whereas Gould [11], and Chetty [5] make it a function of <u>gross</u> investment. Here again it is an empirical matter to decide which of these approaches is more appropriate. For a fuller discussion on this point, see Gould [11].

(6) An excellent discussion on this point is presented in Oi's article [17], and in a recent unpublished paper Donner and Lazar [9] support Oi's contention in a Canadian case.

Oi has pointed out, the cost of labor to the firm includes both variable and fixed elements; for example, variable costs are average hourly wages; fixed costs are hiring and training costs,(7) and these fixed costs can be regarded as investment expenditures for the firm. If one subscribes to the concept of labor as a quasi-fixed factor of production, then it will be reasonable to introduce an adjustment cost function for labor. As in the case of investment, the specification of this function will depend on empirical considerations. As a simple example of this function, let us introduce

$$(2\text{-}4) \qquad W(t) = \bar{w}(t) + \beta J(t) \;, \qquad \beta > 0, \quad \bar{w}(t) > 0$$

where $J(t)$ is an increment of labor defined by

$$L(t+1) = J(t) + L(t).$$

Equation (2-4) satisfies the minimum conditions which the adjustment cost function $D(L) = W(t)L(t)$ has to meet:

$$D(L) > 0, \quad D'(L) > 0 \;, \quad \text{and } D(0) = 0$$

and $\bar{w}(t)$ is analogous to $g(t)$ in equation (2-3), and it is the price that would normally obtain if labor were a perfectly variable factor.

At this stage let us formally state the maximization problem and the dynamics of the system which are explained above:

$$\text{Maximize} \qquad (2\text{-}2) \qquad V = \frac{1}{(1+\rho)^{t_f - t_o}} K(t_f)g(t_f) + \sum_{t=t_o}^{t_f-1} \frac{1}{(1+\rho)^t} R(t)$$

(7) Hiring costs include recruiting, hiring, orientation, terminating, laying off and recalling. Training costs include training, tools and materials, unfilled requisitions, and intracompany transfers. Furthermore, any fringe benefits and supplementary payments to labor will be included in total wage costs.

subject to the following conditions:

(2-1) $$R(t) = P(t)Q(t) - G(t)I(t) - W(t)L(t)$$

(2-3) $$G(t) = g(t) + \alpha I(t)$$

(2-4) $$W(t) = \overline{w}(t) + \beta J(t)$$

(2-5) $$K(t+1) = I(t) + (1 - \sigma)K(t)$$

(2-6) $$L(t+1) = J(t) + L(t)$$

(2-7) $$K(t),\ L(t) > 0$$

Equation (2-5) assumes a constant depreciation rate, σ, for capital sotck. Using Pontryagin maximum principles for the discrete system [18, pp.124-131], we find the optimum trajectory is prescribed by the following requirements:

(2-8) $$H(t) = \frac{1}{(1+\rho)^t} R(t) + \frac{1}{(1+\rho)^t} \lambda(t+1)K(t+1) + \frac{1}{(1+\rho)^t} \mu(t+1)L(t+1)$$

(2-9) $$\begin{bmatrix} K(t+1) \\ L(t+1) \end{bmatrix} = \begin{bmatrix} I(t) + (1 - \sigma)K(t) \\ J(t) + L(t) \end{bmatrix}$$

(2-10) $$\begin{aligned} &- g(t) - 2\alpha I(t) + \lambda(t+1) = 0 \\ &- \beta L(t) + \mu(t+1) = 0 \end{aligned}$$

$$\lambda(t+1) = -\frac{(1+\rho)}{(1-\sigma)} P(t)F_K(t) + \frac{(1+\rho)}{(1-\sigma)} \lambda(t)$$

(2-11)

$$\mu(t+1) = -(1+\rho)P(t)F_L(t) + (1+\rho)W(t) + (1+\rho)\mu(t)$$

$$(2\text{-}12) \quad \begin{bmatrix} \lambda(t_f) \\ \mu(t_f) \end{bmatrix} = \begin{bmatrix} g(t_f) \\ \overline{w}(t_f) \end{bmatrix}$$

where $F_K(t) = \dfrac{\partial F}{\partial K(t)}$, and $F_L(t) = \dfrac{\partial F}{\partial L(t)}$.

(2-8) is the Hamiltonian, and equations in (2-9) are the same as (2-5) and (2-6). Equations in (2-12) are the transversality conditions.(8)

If we solve the set of simultaneous first-order difference equations given by (2-11) for $\lambda(t+1)$ and $\mu(t+1)$ then from (2-10) we can determine $I(t)$ and $L(t)$. However, there are two main difficulties one often faces in

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(8) Conditions (2-11) are similar to (but not the same as) those obtained by using the continuous maximization problem which gives rise to $\dot{\lambda} = -PF_K + (\rho + \sigma)\lambda$, where $\dot{\lambda} = \frac{d\lambda}{dt}$. In equation (2-11) we have assumed that the price, $P(t)$ is independent of $Q(t)$. If the firm faces a downward sloping demand curve then (2-11) should be changed to

$$\lambda(t+1) = -\frac{(1+\rho)}{(1-\sigma)}\left(1 + \frac{1}{\eta(t)}\right)P(t)F_K(t) + \frac{(1+\rho)}{(1-\sigma)}\lambda(t)$$

$$\mu(t+1) = -(1+\rho)\left(1 + \frac{1}{\eta(t)}\right)P(t)F_L(t) + (1+\rho)W(t) + (1+\rho)\mu(t)$$

where $\eta(t) = \dfrac{\partial Q(t)}{\partial P(t)} \dfrac{P(t)}{Q(t)}$.

solving the difference equations, i.e. the split boundary conditions known as the two point boundary value problems (TPBVP) and the fact that the parameters and the form of the function $F[K(t), L(t)]$ are not known.

The first problem of TPBVP exists in our system. For example, the entrepreneour will know the quantities of capital and labor at the initial time t_o, i.e. $K(t_o) = K_o$, and $L(t_o) = L_o$, and furthermore he may expect the transversality conditions (2-12) which simply state that the values of a unit of capital and of a unit of labor are equal to the prices of capital and labor at t_f, respectively. However, the initial conditions of $\lambda(t_o)$ and $\mu(t_o)$ are not known. In short we have split boundary conditions. Obviously what boundary conditions at t_o and t_f are known or not known will vary according to each dynamic problem. However, it is quite likely that in many practical situations, the investigator will face a split boundary problem of one kind or another.

The invariant imbedding procedure [3] is a technique whereby the missing initial (or terminal) conditions are obtained in a direct manner, and thus the original problem is effectively reduced to an initial-value problem. Since how to derive the invariant imbedding procedure is available elsewhere [18, Chapter 15, for example], let us skip here any exposition of the subject. The second problem, i.e., unknown parameters and functional form, is often solved by a sequential estimation method using invariant imbedding equations [7, 19, 20, 5].

To derive the invariant imbedding equations, let us rewrite our system which is given by (2-9) and (2-11) as follows:

$$(2\text{-}9)' \qquad \underline{X}(t+1) = \begin{bmatrix} K(t+1) \\ L(t+1) \end{bmatrix} = \begin{bmatrix} I(t) + (1 - \sigma)K(t) \\ J(t) + L(t) \end{bmatrix} = \underline{f}[\underline{X}(t), \underline{\Lambda}(t), t]$$

$$(2\text{-}11)' \qquad \underline{\Lambda}(t+1) = \begin{bmatrix} \lambda(t+1) \\ \\ \mu(t+1) \end{bmatrix} = \begin{bmatrix} -\frac{(1+\rho)}{(1-\sigma)} P(t)F_K(t) + \frac{(1+\rho)}{(1-\sigma)} \lambda(t) \\ \\ -(1+\rho)P(t)F_L(t) + (1+\rho)W(t) + (1+\rho)\mu(t) \end{bmatrix}$$

$$= \underline{h}[\ \underline{X}(t),\ \underline{\Lambda}(t),\ t]\ .$$

Let us deal with the boundary conditions given as

$$(2\text{-}13) \qquad \underline{X}(t_o) = \begin{bmatrix} K_o \\ L_o \end{bmatrix} = \underline{X}_o \quad \text{, and} \quad \underline{\Lambda}(t_f) = \begin{bmatrix} g(t_f) \\ \\ \overline{w}(t_f) \end{bmatrix} = \underline{\Lambda}_{t_f}\ .$$

The process starts at t_o and ends at t_f. We let

$$(2\text{-}14) \qquad \underline{X}(t_o) = \begin{bmatrix} K_o \\ \\ L_o \end{bmatrix} = \begin{bmatrix} C_1 \\ \\ C_2 \end{bmatrix} = \underline{C}$$

and

$$(2\text{-}15) \qquad \underline{\Lambda}(t_o) = \begin{bmatrix} \lambda(t_o) \\ \\ \mu(t_o) \end{bmatrix} = \begin{bmatrix} r_1(\ \underline{C},\ t_o) \\ \\ r_2(\ \underline{C},\ t_o) \end{bmatrix} = \underline{r}(\ \underline{C},\ t_o)\ .$$

Thus we have imbedded the initial conditions (2-14) in a more general class of initial conditions $\underline{C}$ and we are letting $\underline{r}(\ \underline{C},\ t_o)$ denote the missing initial conditions on $\underline{\Lambda}(t)$ for the process starting at t_o and ending at t_f, and also satisfying $\underline{X}(t_o) = \underline{X}_o$ and $\underline{\Lambda}(t_f) = \underline{\Lambda}_{t_f}$.

Now as usual we make an assumption that $\underline{r}(\underline{C}, t)$ is approximated by a linear function of $\underline{C}$ such that

$$\text{(2-16)} \qquad \underline{r}(\underline{C}, t) = \underline{d}(t) - \underline{U}(t)\underline{C}$$

where

$$\underline{d}(t) = \begin{bmatrix} d_1(t) \\ d_2(t) \end{bmatrix}, \text{ and } \quad \underline{U}(t) = \begin{bmatrix} u_{11}(t) & u_{12}(t) \\ u_{21}(t) & u_{22}(t) \end{bmatrix}.$$

Then we obtain the invariant imbedding equations

$$\text{(2-17)} \qquad \underline{d}(t+1) - \underline{d}(t) - [\underline{U}(t+1) - \underline{U}(t)]\,\underline{C}$$
$$- \underline{U}(t+1)[\,\underline{f}(\underline{C}, \underline{r}, t) - \underline{C}\,] = \underline{h}(\underline{C}, \underline{r}, t) - \underline{r}(\underline{C}, t).$$

In equation (2-11)' above we do not have explicit mathematical functions for the marginal productivities of capital and labor, $F_K(t)$ and $F_L(t)$. If the production function is at least twice differentiable, we may expand $F_K(t)$ and $F_L(t)$ in a Taylor series about $\underline{C} = (K_o, L_o)'$ to obtain

$$\text{(2-18)} \qquad \begin{aligned} F_K(\underline{C}) &= F_K^o + F_{KK}^o(C_1 - K_o) + F_{KL}^o(C_2 - L_o) \\ F_L(\underline{C}) &= F_L^o + F_{LK}^o(C_1 - K_o) + F_{LL}^o(C_2 - L_o) \end{aligned}$$

where $F_i^o = \frac{\partial F}{\partial i}\Big|_{\underline{C}=\underline{X}_o}$ and $F_{ij}^o = \frac{\partial^2 F}{\partial i \partial j}\Big|_{\underline{C}=\underline{X}_o}$, $i,j = K, L$.

As shown in Appendix A, we substitute (2-18) and (2-9)' into (2-17), and equate terms of like powers in $\underline{C}$ to obtain the following Riccati-type difference equations in $d_i(t)$ and $u_{ij}(t)$, $i,j=1,2$:

$$\text{(a)} \quad d_1(t+1) - \frac{1}{2\alpha} u_{11}(t+1)d_1(t+1) - \frac{1+\rho}{1-\sigma} d_1(t)$$
$$+ \frac{1}{2\alpha} g(t)u_{11}(t+1) - \frac{1}{\beta} u_{12}(t+1)\Delta d_2(t+1) + \frac{1+\rho}{1-\sigma} PF_K^o$$
$$- \frac{1+\rho}{1-\sigma} PF_{KK}^o K_o - \frac{1+\rho}{1-\sigma} F_{KL}^o L_o = 0$$

$$\text{(b)} \quad d_2(t+1) - \frac{1}{2\alpha} u_{21}(t+1)d_1(t+1) - (1+\rho)d_2(t)$$
$$+ \frac{1}{2\alpha} g(t)u_{21}(t+1) - \frac{1}{\beta} u_{22}(t+1)\Delta d_2(t+1) + (1+\rho)PF_L^o$$
$$- (1+\rho)PF_{KL}^o K_o - (1+\rho)PF_{LL}^o L_o - (1+\rho)W(t) = 0$$

$$\text{(c)} \quad -(1-\sigma)u_{11}(t+1) + \frac{1}{2\alpha} [u_{11}(t+1)]^2 + \frac{1}{\beta} u_{12}(t+1)\Delta u_{21}(t+1)$$
$$+ \frac{1+\rho}{1-\sigma} u_{11}(t) + \frac{1+\rho}{1-\sigma} PF_{KK}^o = 0$$

(2-19)

$$\text{(d)} \quad -(1-\sigma)u_{21}(t+1) + (1+\rho)u_{21}(t+1)u_{21}(t) + \frac{1}{2\alpha} u_{21}(t+1)u_{11}(t+1)$$
$$+ \frac{1}{\beta} u_{22}(t+1)\Delta u_{21}(t+1) + (1+\rho) PF_{KL}^o = 0$$

$$\text{(e)} \quad - u_{12}(t+1) + \frac{1+\rho}{1-\sigma} u_{12}(t) + \frac{1}{2\alpha} u_{11}(t+1)u_{12}(t+1)$$
$$+ \frac{1}{\beta} u_{12}(t+1)\Delta u_{22}(t+1) + \frac{1+\rho}{1-\sigma} PF_{KL}^o = 0$$

$$\text{(f)} \quad - u_{22}(t+1) + (1+\rho)u_{22}(t) + \frac{1}{2\alpha} u_{21}(t+1)u_{12}(t+1)$$
$$+ \frac{1}{\beta} u_{22}(t+1)\Delta u_{22}(t+1) + (1+\rho)PF_{LL}^o = 0$$

where $\Delta d_2(t+1) = d_2(t+1)$, and $\Delta u_{2j} = u_{2j}(t+2) - u_{2j}(t+1)$, $j = 1,2$.
The six equations in (2-19) can be solved backwards starting from the

terminal conditions:

$$\underline{\Lambda}(t_f) = \begin{bmatrix} \lambda(t_f) \\ \\ \mu(t_f) \end{bmatrix} = \begin{bmatrix} g(t_f) \\ \\ \overline{w}(t_f) \end{bmatrix}.$$

From (2-15) and (2-16) we see that

$$\text{(2-20)} \qquad \underline{\Lambda}(t_f) = \begin{bmatrix} \lambda(t) \\ \\ \mu(t) \end{bmatrix} = \begin{bmatrix} r_1(\underline{C},t) \\ \\ r_2(\underline{C},t) \end{bmatrix} = \begin{bmatrix} d_1(t)-u_{11}(t)C_1- u_{12}(t)C_2 \\ \\ d_2(t) -u_{21}(t)C_1 -u_{22}(t)C_2 \end{bmatrix}.$$

Then at $t=t_f$ we must have

$$\begin{bmatrix} d_1(t) = g(t_f) \\ \\ d_2(t) = \overline{w}(t_f) \end{bmatrix} \quad \text{and} \quad u_{ij}(t_f) = 0 \quad \text{for } i, j =1,2.$$

At $t=t_f-1$, we use $u_{ij}(t_f) = 0$ to obtain new values for $u_{ij}(t_f-1)$, $i,j=1,2$ from equations (c)-(f) in (2-19). Then in turn from equations (a) and (b) in (2-19) we can solve for $d_1(t_f-1)$ and $d_2(t_f-1)$. Then we solve for $\lambda(t_f-1)$ and $\mu(t_f-1)$ from (2-20) noting that $C_1 \doteqdot K_o$, and $C_2 \doteqdot L_o$. Once these are solved then from the equations in (2-10) we can find values for $I(t_f-2)$ and $L(t_f-2)$.

To solve for $I(t)$ and $L(t)$ in the manner explained above we are required to know the values of the parameters, α, β, σ, and ρ, and the values of $P(t)$, $F_i^o(t)$ $(i=K, L)$ and $F_{ij}^o(t)$ $(i,j =K, L)$. If some of the parameters, say, α and β, are unknown, then we may treat them as the "state variables" such that $\alpha(t+1) = \alpha(t)$ and $\beta(t+1) = \beta(t)$, and by incorporating them into our system [i.e. (2-9)'] we may develop "on-line" estimation of states and parameters by applying the least squares criteria as the cost function. Examples of on-line estimation of states and parameters for simple discrete dynamic systems are available [19, 20]. However, if the number of unknown parameters to be estimated is large, one may not easily obtain stable estimates of states and parameters. Furthermore, as the order and nonlinearity of the system are increased, the stability of the estimates becomes more difficult to ensure.

Usually we do not know the mathematical form of the production function $F(K,L)$, and the values of $P(t)$, $t \in [t_o , t_f]$ may not be easily predicted. As we see in the succeeding discussion, however, the estimation of states and parameters could be considerably simplified and could be handled by a simultaneous estimation method which is available within the usual domain of econometric methods. This simplification hinges upon the assumption that we can treat $F_{ij} = 0$, $i, j = K, L$ for all $t \in [t_o, t_f]$.

If $F_{KK} = F_{KL} = F_{LL} = 0$ for all $t \in [t_o, t_f]$, then given the conditions that $u_{ij}(t_f)=0$ for $i,j =1,2$, we will easily see from equations (c)-(f) in (2-19) that $u_{ij}(t) = 0$ for all $t \in [t_o, t_f]$. Since $\underline{\Lambda}(t) = \underline{r}(\underline{C}, t)$ for all $t \in [t_o, t_f]$, equations (a) and (b) in (2-19) become

$$\text{(2-21.a)} \quad d_1(t+1) - \frac{1+\rho}{1-\sigma} d_1(t) + \frac{1+\rho}{1-\sigma} PF_K = 0$$

$$\text{(2-21.b)} \quad d_2(t+1) - (1+\rho)d_2(t) + (1+\rho)PF_L + (1+\rho)W(t) = 0$$

or the same as the equations in (2-11) with $\lambda(t) = d_1(t)$ and $\mu(t) = d_2(t)$. Given the terminal conditions

$$\underline{\Lambda}(t_f) = \begin{bmatrix} \lambda(t_f) \\ \mu(t_f) \end{bmatrix} = \begin{bmatrix} g(t_f) \\ \overline{w}(t_f) \end{bmatrix}$$

again equations (2-21.a) and (2-21.b) can be solved backwards to yield

$$\text{(2-22.a)} \quad \lambda(t) = \left[\frac{1-\sigma}{1+\rho}\right]^{t_f-t} g(t_f) + \sum_{\tau=t}^{t_f-1}\left[\frac{1-\sigma}{1+\rho}\right]^{\tau-t} P(\tau)F_K(\tau)$$

$$\text{(2-22.b)} \quad \mu(t) = \left[\frac{1}{1+\rho}\right]^{t_f-1} \bar{w}(t_f) + \sum_{\tau=t}^{t_f-1}\left[\frac{1}{1+\rho}\right]^{\tau-t} [P(\tau)F_L(\tau) - W(\tau)].$$

Then substituting these equations into those in (2-10) we find

$$\text{(2-23.a)} \quad I(t) = \frac{1}{2\alpha}\{\left[\frac{1-\sigma}{1+\rho}\right]^{t_f-t-1} g(t_f) + \sum_{\tau=t+1}^{t_f-1}\left[\frac{1-\sigma}{1+\rho}\right]^{\tau-t-1} P(\tau)F_K(\tau) - g(t)\}$$

$$\text{(2-23.b)} \quad L(t) = \frac{1}{\beta}\{\left[\frac{1}{1+\rho}\right]^{t_f-t-1} \bar{w}(t_f) + \sum_{\tau=t+1}^{t_f-1}\left[\frac{1}{1+\rho}\right]^{\tau-t-1} [P(\tau)F_L(\tau) - W(\tau)]\}.$$

These equations can be used for the estimation of α, β, σ, and ρ. However, they can be further simplified if we subtract from each of these equations $\frac{1-\sigma}{1+\rho} I(t+1)$ and $\frac{1}{1+\rho} L(t+1)$ respectively:

$$\text{(2-24.a)} \quad I(t) - \frac{1-\sigma}{1+\rho} I(t+1) = \frac{1}{2\alpha}\{P(t+1)F_K(t+1)$$

$$- [g(t) - \frac{1-\sigma}{1+\rho} g(t+1)]\} + \varepsilon_{1,t} - \frac{1-\sigma}{1+\rho} \varepsilon_{1,t+1}$$

$$(2\text{-}24.b) \quad L(t) - \frac{1}{1+\rho} L(t+1) = \frac{1}{\beta} [P(t+1)F_L(t+1)$$

$$- W(t+1)] + \varepsilon_{2,t} - \frac{1}{1+\rho} \varepsilon_{2,t+1}$$

where $\varepsilon_{1,t}$ and $\varepsilon_{2,t}$ are the distrubance terms attached in equations (2-23.a) and (2-23.b).

We see that the variable $g(t)$ in equation (2-24.a) is not observable if we recall equation (2-3) and the discussion following it. However, from equation (2-3) we have $g(t) = G(t) - \alpha I(t)$ and by substituting this into (2-24.a) and rearranging the terms we obtain

$$(2\text{-}24.a)' \quad I(t) - \frac{1-\sigma}{1+\rho} I(t+1) = \frac{1}{\alpha}\{ P(t+1)F_K(t+1)$$

$$- [G(t) - \frac{1-\sigma}{1+\rho} G(t+1)]\}$$

$$+ 2(\varepsilon_{1,t} - \frac{1-\sigma}{1+\rho} \varepsilon_{1,t+1}).$$

III. The Estimation of the Model

Equations (2-24.a)' and (2-24.b) are, respectively, the investment and labor demand equations which can be estimated by an appropriate simultaneous estimation method. The simultaneity problem will arise if the variables on the right hand side such as P, F_K, G, F_L, and W cannot be treated as exogenous variables (in the statistical sense). Even if they were exogenous the simultaneity problem arises, if the variables on the left hand side $\frac{1-\sigma}{1+\rho}$ I(t+1) and $\frac{1}{1+\rho}$ L(t+1) were to be transferred to the right hand side of these equations to estimate $\frac{1-\sigma}{1+\rho}$ and $\frac{1}{1+\rho}$.

Equations (2-24.a)' and (2.24.b) involve disturbance terms which are first order moving average processes, and thus we face the problem of autocorrelation within the context of simultaneous equations. If the "endogenous variables" in these equations such as PF_K, G, PF_L, and W have well defined reduced form equations in a linear simultaneous equation system, then a procedure such as [2] may be applied, with modification, to our system.(9) However, it is not likely that these endogenous variables possess well defined reduced form equations, and thus let us resort to an instrumental variables estimation method which will yield at least a consistent (but not necessarily efficient) estimator. We start with the investment equation (2-24.a)'. Suppose that the disturbance term $\varepsilon_{1,t}$ obeys the following conditions:

$$(3\text{-}1) \qquad E(\varepsilon_{1,t}) = 0, \quad E(\varepsilon_{1,t}\,\varepsilon_{1,t'}) = \delta_{tt'}\,\sigma_1^2 \text{ , for all } t$$

where $\delta_{tt'}$ is the Kronecker delta. Then the transformed disturbance term

$$(3\text{-}2) \qquad e_{1,t} = 2(\varepsilon_{1,t} - \kappa_1\varepsilon_{1,t+1}) \quad , \quad \kappa_1 = \frac{1-\sigma}{1+\rho}$$

obeys

$$E(e_{1,t}) = 0, \quad \text{for all } t$$

$$E(e_{1,t}^2) = 4(1+\kappa_1^2)\,\sigma_1^2 \quad \text{, for all } t$$

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(9) Alternatively, if we assume that $\varepsilon_{1,t}$ and $\varepsilon_{2,t}$ are multinormally distributed, we may develop a full information maximum likelihood (FIML) procedure. Here we do not pursue this since the FIML procedure tends to be computationally more cumbersome than a single equation method.

$$E(e_{1,t}\, e_{1,s}) = \begin{cases} -4\kappa_1\sigma_1^2 & \text{for } t, s \text{ such that } |s-t| = 1 \\ 0 & \text{for } t, s \text{ such that } |s-t| \geqq 2 \end{cases}$$

i.e. the variance-covariance matrix of $e_{1,t}$ becomes

$$\text{(3-3)} \qquad \mathrm{Cov}(e_1) = 4\sigma_1^2 \begin{bmatrix} 1+\kappa_1^2 & -\kappa_1 & 0 & \cdot & \cdot & \cdot & \cdot & 0 \\ -\kappa_1 & 1+\kappa_1^2 & -\kappa_1 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & & \cdot & & & & & \vdots \\ \cdot & & & \cdot & & & & \vdots \\ \cdot & & & & & \cdot & & -\kappa \\ 0 & \cdot & & & \cdot & \cdot & -\kappa_1 & 1+\kappa_1^2 \end{bmatrix} = V .$$

Let us rewrite equation (2-24.a)' as

$$\text{(3-4)} \qquad y^* = \gamma_1 x_1^* + e_1$$

where

$$y^* = \begin{bmatrix} I(1) - \kappa_1 I(2) \\ \vdots \\ I(T) - \kappa_1 I(T+1) \end{bmatrix}, \quad \gamma_1 = \frac{1}{\alpha}, \quad x_1^* = \begin{bmatrix} P(2)F_K(2) - [G(1) - \kappa_1(G(2)] \\ \vdots \\ P(T+1)F_K(T+1) - [G(T) - \kappa_1 G(T+1)] \end{bmatrix}$$

and

$$e_1 = \begin{bmatrix} e_{11} \\ \vdots \\ e_{1T} \end{bmatrix} .$$

If we can find an instrumental variable $z_1 = (z_{11}, \ldots, z_{1T})'$ such that $\operatorname*{plim}_{T\to\infty}\left(\frac{z_1' x_1^*}{T}\right) \neq 0$ and finite, and $\operatorname*{plim}_{T\to\infty}\left(\frac{z_1' e_1}{T}\right) = 0$, then the estimator of γ_1 can be given as

$$(3\text{-}5) \qquad \hat{\gamma}_1 = (z_1'x_1^*)^{-1}z_1'y^*$$

which will be consistent for a given κ_1, and an estimate of the asymptotic variance of $\hat{\gamma}_1$ is given by

$$(3\text{-}6) \qquad \mathrm{Var}(\hat{\gamma}_1) = (z_1'x_1^*)^{-1}(z_1'Vz_1)(x_1^{*\prime}z_1)^{-1}$$

Equation (3-4) contains a parameter $\kappa_1 = \frac{1-\sigma}{1+\rho}$. If its value is not available then the estimate of κ_1, $\hat{\kappa}_1$, may be made in either of the following two ways:

(1) <u>Iteration</u>: First start with an initial value of $\hat{\kappa}_1 = \kappa_1^o$, and using equation (3-5) estimate $\hat{\gamma}_1$; then estimate another $\hat{\kappa}_1$ from

$$(3\text{-}7) \qquad \hat{\kappa}_1 = \frac{\sum_{t=1}^{T} \{I(t) - \hat{\gamma}_1[P(t+1)F_K(t+1) - G(t)]\}\, z_2(t)}{\sum_{t=1}^{T} [I(t+1) + \hat{\gamma}_1 G(t+1)]\, z_2(t)}$$

where $z_2(t)$ is an instrumental variable such that $\underset{T\to\infty}{\mathrm{plim}} \left(\frac{1}{T} \sum_{t=1}^{T} e_{1t} z_2(t)\right) = 0$ and $\underset{T\to\infty}{\mathrm{plim}}\left(\frac{1}{T}\sum_{t=1}^{T} [I(t+1) + \hat{\gamma}_1 G(t+1)]\, z_2(t)\right) \neq 0$. Then iterate the estimation until $\hat{\kappa}_1$ converges. It is easy to show that the converged value of $\hat{\gamma}_1$ and $\hat{\kappa}_1$ are consistent.

(2) Scanning: Since $0 < \kappa_1 < 1$, give the values of κ_1, say, $\kappa_1^1, \kappa_1^2, \ldots, \kappa_1^m$ in the interval of $(0, 1)$, and for each κ_1^i estimate $\hat{\gamma}_1$ by by (3-5). Choose $\hat{\gamma}_1(\kappa_1^i)$ such that the estimate of $\mathrm{var}(e_{1,t})$ attains a minimum.

If the instrumental variable $z_2(t)$ is not readily available the iteration algorithm does not yield a consistent estimate of κ_1. [$\hat{\gamma}_1$ will be consistent as long as one has z_1 and $\mathrm{plim}\ \hat{\kappa}_1 < \infty$]. The point of convergence of $\hat{\kappa}_1$ may depend upon the selection of the initial value, κ_1^o. The difficulty with the scanning algorithm above is that there is no guarantee that the estimate of $\mathrm{Var}(e_{1,t})$ attains a minimum within $\kappa_1^i \in [0, 1]$.

Estimation of the paramters, β and $\kappa_2' = \frac{1}{1+\rho}$ in equation (2-24.b) can be made in the same fashion as that of α and κ_1 in equation (2-24.a)', except that for the algorithm (1) equation (3-7) is replaced by

$$(3\text{-}8) \qquad \hat{\kappa}_2 = \frac{\sum_{t=1}^{T} \{L(t) - \hat{\gamma}_2[P(t+1)F_L(t+1) - W(t+1)]\}\ z_3(t)}{\sum_{t=1}^{T} L(t+1)\ z_3(t)}$$

where $\gamma_2 = \frac{1}{\beta}$, and $z_3(t)$ is an instrumental variable such that

$\underset{T\to\infty}{\mathrm{plim}}(\frac{1}{T} \sum_{t=1}^{T} L(t+1) z_3(t)) \neq 0$ and finite, and $\underset{T\to\infty}{\mathrm{plim}}(\frac{1}{T} \sum_{t=1}^{T} e_{2,t} z_3(t)) = 0.$

<u>An Illustration of the Estimation (Total Canadian Manufacturing Industries)</u>: The model developed in the previous section is based on the theory of the firm which follows the maximization problem described by equation (2-2) subject to the system given in equations (2-1)- (2-7), and hence an empirical test of the model using the estimation methods given above will best be done if we have data of a firm whose behavior fits with our system. In the absence of such data, let us illustrate the estimation by using annual data on the total Canadian manufacturing industries from 1947 to 1969;(10) we are aware of aggregation problem which may make a test of the model obscure.

In equations (2-24.a)' and (2-24.b) we have the marginal physical products of capital and labor, F_K and F_L as the explanatory variables. In the absence of an explicit mathematical form of the production function, we use the average productivities, Q(t)/K(t), and Q(t)/L(t), as the proxies for these variables on the following two grounds: (1) in many practical cases entrepreneurs and labor leaders tend to rely on average productivities as the measures of productivity performance, and (2) if the production function is a Cobb-Douglas or a CES with constant returns to scale as many empirical studies assume then the approximation of the marginal productivities by the average productivities may be reasonable.(11)

<u>The Choice of Instrumental Variables</u>: As instrumental variables we use data on the United States total manufacturing industries, since the Canadian manufacturing industries tend to be dominated by U.S. subsidiaries. The causal flow is from the United States parent companies to their Canadian counterparts. Consequently U.S. data may satisfy the two conditions required for instrumental variables. Rather than using U.S. data directly we have used the values computed from the regression of Canadian variables on their U.S. counterparts (regression results are given in Appendix B).

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(10) Sources of data are given in Appendix B.

(11) A CES production function with constant returns to scale gives rise to $F_K = h_1\left(\frac{Q}{K}\right)^{1/\sigma}$ and $F_L = h_2\left(\frac{Q}{L}\right)^{1/\sigma}$, where h_1, h_2, and σ are parameters. $\sigma = 1$ is the case of Cobb-Douglas.

The instrumental variable z_1 in equation (3-5) for the estimation of the parameters in the investment equation (2-24.a)' is represented by $\left[\widehat{\frac{P(t)Q(t)}{K(t)}}\right] - [\hat{G}(t) - \hat{G}(t+1)]$, where "^" indicates computed values. For $z_2(t)$ in equation (3-7) we have used $\hat{I}(t+1) + \hat{G}(t+1)$. For the estimation of the parameters in the labor equation (2-24.b), z_1 is given by $\left[\widehat{\frac{P(t)Q(t)}{L(t)}}\right]$ and $z_3(t)$ in equation (3-8) is given by $\hat{L}(t+1)$.

Tables 3.1 and 3.2 present estimation results using the scanning algorithm. The $\bar{R}^2$ and DW denote, respectively the coefficient of determination adjusted for degrees of freedom and the Durbin-Watson test statistic, and the figure in parentheses below the estimated coefficient is its estimated standard error. For these two cases we have found minimum points of the

Table 3.1 The Estimated Parameters of Equation (2-24.a)' by the Scanning Algorithm

κ_1	$(1/\alpha)$	$Var(e_{1,t})$	$\bar{R}^2$	DW
.895	139.5717 (42.6195)	40810	.523	1.55
.900	129.6145 (42.2783)	40794	.523	1.55
.905	119.8022 (41.9461)	40784	.522	1.55
.910	110.1318 (41.6230)	40781	.522	1.56
.915	100.6003 (41.3083)	40785	.522	1.56
.920	91.2049 (41.0020)	40794	.522	1.56
.925	81.9426 (40.7037)	40810	.522	1.56

Table 3.2 The Estimated Parameters of Equation (2-24.b) by the Scanning Algorithm

κ_2	Constant Term	$(1/\beta)$	$Var(e_{2,t})$	$\bar{R}^2$	DW
.910	54.5015 (46.5295)	21.0721 (10.7553)	1481.54	.712	1.70
.915	50.7806 (46.7049)	19.6433 (10.7507)	1479.85	.708	1.70
.920	47.6026 (46.8857)	18.2142 (10.7472)	1479.39	.703	1.72
.925	43.3380 (47.0728)	16.7860 (10.7447)	1479.17	.700	1.72
.930	39.6170 (47.2646)	15.3572 (10.7430)	1479.18	.697	1.73
.935	35.8960 (47.4629)	13.9284 (10.7424)	1479.43	.695	1.73
.940	32.1750 (47.6656)	12.4996 (10.7426)	1479.91	.690	1.73

estimated $Var(e_{i,t})$, i=1, 2 within the scanned range, R, of κ_1 and κ_2: R = [.5, 1.0]. For the investment equation the minimum estimate of $Var(e_{1,t})$ is at $\kappa_1 = .910$ and for the labor equation it is at $\kappa_2 = .925$. A constant term was added to equation (2-24.b)' due to the fact that the variance estimate [$Var(e_{2,t}$] was much smaller when the constant term was added to the equation, while this was not the case for the investment equation.

The estimation results using the iteration algorithm are given in Table 3.3. The criterion for the convergence is given at

$$\left| \frac{\kappa_i^{(k)} - \kappa_i^{(k-1)}}{\kappa_i^{(k-1)}} \right| < .0005 ,$$ where $\kappa_i^{(k)}$ (i=1,2) indicates the k-th

iteration of κ_i. The converged values of κ_i (i=1,2) were close to the estimates of κ_i by the scanning algorithm. The point of convergence of $\kappa_i^{(k)}$ may depend on the choice of the initial value, κ_i^o. We experimented with several initial values for κ_i^o, and found that κ_2 of the labor equation

Table 3.3 The Estimated Parameters of Equations (2-24.a)' and (2-24.b) by the Iteration Algorithm

Investment Equation				
κ_1	$(1/\alpha)$	$\mathrm{Var}(e_{1,t})$	$\bar{R}^2$	DW
.9118	107.6052 (41.5392)	40781	.522	1.56

Labor Equation					
κ_2	Constant Term	$(1/\beta)$	$\mathrm{Var}(e_{2,t})$	$\bar{R}^2$	DW
.9242	43.9151 (47.0440)	17.0068 (10.7450)	1479.19	.700	1.72

tended to be quite sensitive to the choice of κ_2^o when κ_2^o is in the range of .88 to .95. This may be due to the fact that as shown in Table 3.2, the estimate of $\mathrm{Var}(e_{2,t})$ is very "flat" for $\kappa_2 \in [.88, .95]$ varying between 1479 and 1485. This difficulty was not encountered in the investment equation.

IV. Conclusions

Given our system described in equations from (2-2) to (2-7) the equations (2-24.a)' and (2-24.b) are derived on the assumption that $F_{ij} = 0$ $(i,j = K, L)$ for all $t \in [t_o, t_f]$. If a mathematical form of the production function is known or if the values of F_i, and F_{ij} $(i,j = K, L)$ are known, then it will be preferable that the estimation of the states and parameters is made by the use of the sequential estimation method as given in [19, 20]. However, the on-line estimation may not yield stable estimates if the number of unknown parameters to be estimated is large. It may be interesting to conduct a controlled experiment to see how the estimates of states and parameters with an assumption of $F_{ij} = 0$ may perform against those of the sequential estimation procedure. Obviously, results will depend on the dynamics and mathematical form of the system, the number of unknown parameters, and the nature of state variables, i.e., whether they are subject to noises.

The equations to be estimated, (2-24.a)' and (2-24.b) are based on the boundary conditions specified in (2-13) and on the assumption that $\underline{r}(\underline{C}, t)$ can be approximated by a linear function of $\underline{C}$ [See equation (2-16)]. Different boundary conditions will give rise to invariant imbedding equations different from those given in (2-19), and the formulation of a dynamic system will vary according to each case one studies. The total Canadian manufacturing industries are used only as an illustration of the estimation of the model derived from our system in (2-2) to (2-7). If any equations of the system change, for example, if the cost of adjustment functions (2-3) and (2-4) are altered, then the model derived above has to be modified accordingly.

Appendix A Derivation of Equations in (2-19)

We substitute (2-18) and (2-9)' into (2-17) to obtain

$$(A-1)\quad \begin{bmatrix} (a) \\ (b) \end{bmatrix} = \begin{bmatrix} d_1(t+1) - d_1(t) \\ d_2(t+1) - d_2(t) \end{bmatrix} - \begin{bmatrix} u_{11}(t+1)C_1 - u_{11}(t)C_1 + \\ u_{21}(t+1)C_1 - u_{21}(t)C_1 + \end{bmatrix}$$

$$\begin{matrix} u_{12}(t+1)C_2 - u_{12}(t)C_2 \\ u_{22}(t+1)C_2 - u_{22}(t)C_2 \end{matrix} \Bigg] - \begin{bmatrix} u_{11}(t+1)[I(t)-\sigma C_1] + u_{12}(t+1)J(t) \\ u_{21}(t+1)[I(t)-\sigma C_1] + u_{22}(t+1)J(t) \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{1+\rho}{1-\sigma}PF_K^o - \frac{1+\rho}{1-\sigma}PF_{KK}^o(C_1-K_o) - \frac{1+\rho}{1-\sigma}PF_{KL}^o(C_2-L_o) + \frac{1+\rho}{1-\sigma}r_1(\underline{C}, t) \\ -(1+\rho)PF_L^o - (1+\rho)PF_{KL}^o(C_1-K_o) - (1+\rho)PF_{LL}^o(C_2-L_o) + (1+\rho)W(t) + r_2(\underline{C}, t) \end{bmatrix}$$

From equations in (2-10) we have

$$I(t) = \frac{\lambda(t+1) - g(t)}{2\alpha} \qquad \text{and} \qquad L(t) = \frac{\mu(t+1)}{\beta} .$$

Recall (2-15) imply $\lambda(t+1) = r_1(\underline{C}, t+1)$, and $\mu(t+1) = r_2(\underline{C}, t+1)$. Hence $I(t)$ and $J(t)$ in (A-1) above become

$$(A-2)\qquad I(t) = \frac{r_1(\underline{C}, t+1) - g(t)}{2\alpha}$$

$$J(t) = L(t+1) - L(t) = \frac{\mu(t+2)}{\beta} - \frac{\mu(t+1)}{\beta}$$

$$= \frac{r_2(\underline{C}, t+2)}{\beta} - \frac{r_2(\underline{C}, t+1)}{\beta}$$

Equation (2-16) is expressed as

$$(A\text{-}3)\qquad \begin{aligned} r_1(\underline{C}, t) &= d_1(t) - u_{11}(t)C_1 - u_{12}(t)C_2 \\ r_2(\underline{C}, t) &= d_2(t) - u_{21}(t)C_1 - u_{22}(t)C_2 \end{aligned} \quad .$$

We substitute (A-2) and (A-3) into (A-1) and equate terms of like wise powers in C (i.e. constant term, C_1, and C_2 in (A-1.a) and (A-1.b)) to obtain the set of equations in (2-19).

Appendix B

Sources of Canadian Data: Gross fixed capital formation, I(t), net stock of fixed capital, K(t), and the price of capital goods, G(t), were taken from Dominion Bureau of Statistics, Fixed Capital Flows and Stocks Manufacturing, Canada 1926-1960, Catalogue No. 13-523, August 1966; for 1961-1969, the figures were provided by the D.B.S. Gross domestic product of manufacturing, P(t)Q(t), were taken from National Accounts Income and Expenditure, DBS Catalogue No.13-201, various issues. Employment, L(t), was taken from Labor Force, DBS Catalogue No. 71-001, various issues. Wages, salaries and supplements, W(t)L(t), were taken from the same sources as gross domestic product of manufacturing data.

Sources of U.S. Data: U.S. manufacturing capital stock, $K(t)^{US}$, was taken from, "Fixed Business Capital in the United States, 1925-1965," Survey of Current Business, December 1967 issue,for 1946-1966. The series for 1967-1969 was extended on the basis of gross private investment series given in National Income Accounts. Wage rate, $W(t)^{US}$, employment, $L(t)^{US}$, gross investment, $I(t)^{US}$, and the price of capital goods, $G(t)^{US}$, were taken from U.S. Department of Commerce, The National Income and Product Accounts of the United States, 1929-1965, Statistical Tables, August 1966 and the Survey of Current Business, various issues.

<u>Regression of Canadian Data on U.S. Data</u>: The $\bar{R}^2$, DW, denote, respectively, the coefficient of determination adjusted for degrees of freedom and the Durbin-Watson test statistic. The figure in parentheses right below each estimated coefficient is its estimated standard error.

(B-1) $$I(t) = \underset{(.0022)}{.0192}\ I(t)^{US} - \underset{(172.0731)}{215.3650} \qquad \bar{R}^2 = .76 \quad DW = 1.40$$

(B-2) $$\frac{P(t)Q(t)}{K(t)} = \underset{(.0332)}{.4176} \left[\frac{P(t)Q(t)}{K(t)}\right]^{US} + \underset{(.0500)}{.1581} \qquad \bar{R}^2 = .87 \quad DW = .74$$

(B-3) $$G(t) = \underset{(.0638)}{1.5087}\ G(t)^{US} - \underset{(.0617)}{.4236} \qquad \bar{R}^2 = .96 \quad DW = .37$$

(B-4) $$L(t) = \underset{(.0075)}{.1091}\ L(t)^{US} - \underset{(127.5106)}{493.1340} \qquad \bar{R}^2 = .90 \quad DW = .97$$

(B-5) $$\frac{P(t)Q(t)}{L(t)} = \underset{(.0244)}{.9642} \left[\frac{P(t)Q(t)}{L(t)}\right]^{US} - \underset{(.1884)}{.6200} \qquad \bar{R}^2 = .99 \quad DW = 1.00$$

(B-6) $$W(t) = \underset{(.0233)}{1.0040}\ W(t)^{US} - \underset{(.1248)}{.8656} \qquad \bar{R}^2 = .99 \quad DW = .61$$

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